A game theory model for currency markets stabilization

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Abstract

The aim of this paper is to propose a methodology to stabilize the currency markets by adopting Game Theory. Our idea is to save the Euro from the speculative attacks (due the crisis of the Euro-area States), and this goal is reached by the introduction, by the normative authority, of a financial transactions tax. Specifically, we focus on two economic operators: a real economic subject (as for example the Ferrari S.p.A., our first player), and a financial institute of investment (the Unicredit Bank, our second player). The unique solution which allows both players to win something, and therefore the only one collectively desirable, is represented by an agreement between the two subjects. So the Ferrari artificially causes an inconsistency between currency spot and futures markets, and the Unicredit takes the opportunity to win the maximum possible collective sum, which later will be divided with the Ferrari by contract.

Keywords: Currency Markets, Financial Risk, Financial Crisis, Game Theory, Speculation
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1. Introduction

The recent financial crisis has shown that, in order to stabilize markets, it is not enough to prohibit or to restrict short-selling. In fact:

big speculators can influence badly the market and take huge advantage from arbitrage opportunities, caused by themselves.

For nearly eight years from January 2001, Euro has had a upward trend versus the U.S. Dollar and in April 2008 Euro peaked out at 1.6 a U. S. Dollar. But after this date, Euro has declined by 17% until March 2012 (see the figure 1 [see also [13]]).

Figure 1: U.S. Dollar-Euro exchange rate.

This decrease of the Euro value is due to the crisis that has hit the States of Euro-area and to the uncertain conditions of recovery of European economies. Moreover, the recent developments in the Greek crisis, which could lead to an exit of Greece from the Euro, certainly do not help the Euro against speculative attacks. So, a further decrease in the Euro value would make even more complicated the economic situation in Europe.

In this paper, by the introduction of a tax on financial transactions, we propose (using Game Theory [for a complete study of a game see [1, 2, 3, 5, 6, 7, 8, 9, 10, 11, 12]]) a method aiming to limit the Euro speculations of medium and big financial operators and, consequently, a way to make more stable the currency markets. Moreover, our aim is attained without inhibiting the possibilities of profits. At this purpose, we will present and study a natural and quite general normal form game - as a possible standard model of fair interaction between two financial operators - which gives to both players mutual economic advantages.

As our first player we choose the Ferrari as an exemplary multinational enterprise. The Ferrari is a big economic subject that is famous throughout the world (everyone dreams to can drive a Ferrari car) and has a huge turnover. In fact, the Ferrari, despite being of Italian origin, is now established in all 5 continents of the Earth and is a multinational corporation in every respect. For this reason, the Ferrari is often exposed to currency risk. But the ordinary activities of the Ferrari is to sell luxury cars, not to act on the currency market paying attention to the fluctuations of the currency values. So, taking in account only the 2010, the Ferrari has spent the pharaonic sum of 885 million Euros for the conclusion of derivative contracts for hedging against currency risk (these data are readily available on the financial statements of the Ferrari).

As our second player we choose the Unicredit Bank because it is one of the main financial institute of the world and it acts constantly on the financial markets.

1.1. Financial preliminaries

Here, we recall the financial concepts that we shall use in the present article.

1)Any (positive) real number is a ***(proper) purchasing strategy***; a negative real number is a ***selling strategy***.

2) The ***spot market*** is the market where it is possible to buy and sell at current prices.

3) ***Futures*** are contracts between two parties to exchange, for a price agreed today, a specified quantity of the underlying commodity, at the expiry of the contract.

4) In derivatives market there are three main ***categories of operators***, depending on the purpose with which use the derivative contract: hedgers, speculators and arbitrageurs.

4.1. ***Hedgers*** use forwards and futures to reduce the risks resulting from their exposures to market variables. Forward hedges eliminate the uncertainty on the price to pay for the purchase (or receivable for the sale) of the underlying asset, but not necessarily lead to a better result. The use of the derivative allows to neutralize the adverse trend of the market, offsetting losses/gains on the price of the underlying asset with the gains/losses obtained on the derivatives market.

4.2. ***Speculators*** realize investment strategies, buying (or selling) futures and then sell (or buy) them at a price higher (or lower). Who decides to speculate assumes a risk about the favorable or unfavorable trend of the futures market. The futures market offers a financial leverage to speculators, which are able to take relatively large positions with a low initial outlay.

4.3. ***Arbitrageurs*** take the offsetting positions of two or more contracts to lock in a risk-free profit, and take advantage of a price difference between two or more markets. The arbitrageurs exploit a temporary mismatch between the performance (intended to coincide when the contract expires) of the futures market and the underlying market.

5) ***A hedging operation*** through futures consists in purchase of futures contracts, in order to reduce exposure to specific risks on market variables (in this case on the price). In practice, the loss potential that is obtained on the spot market (the market at current prices) was offset by the gain on futures contracts.

6) A hedging operation is said ***perfect*** when it completely eliminates the risk of the case.

7) ***The futures price*** is linked to the underlying spot price. We assume that:

7.1. the underlying commodity does not offer dividends;

7.2. the underlying commodity hasn't storage costs and has not convenience yield to take physical possession of the goods rather than futures contract.

8)The general relationship linking the futures price F_t, with delivery time T, and spot price S_t, with sole interest capitalization at the time T, is $F_t = S_t u^T$, where $u = 1 + i$ is the capitalization factor of the futures and i the corresponding interest rate. If not, the arbitrageurs would act on the market until futures and spot prices return to levels indicated by the above relation.

1.2. Methodologies

The strategic game G, we propose for modeling our financial interaction, requires a construction on 3 times, say time 0, 1 and 2.

0) At time 0, the Ferrari knows the quantity of his U. S. Dollar financial credits that derive from the sale of cars. It can choose to buy Euro futures contracts in order to hedge the currency risk on its no-Euro financial credits.

1) At time 1, on the other hand, the Unicredit acts with speculative purposes on the currency spot markets (buying or short-selling Euros at time 0) and on the currency futures market (by the opposite action of that performed on the spot market). The Unicredit may so take advantage of the temporary misalignment of the Euro spot and futures prices (expressed in U.S. Dollars), created by the hedging strategy of the Ferrari.

2) At the time 2, the Unicredit will cash or pay the sum determined by its behavior in the futures market at time 1.

Remark. In this game, we suppose that the no-Euro credits of the Ferrari are U.S. Dollar credits, but this game theory model is also valid for any currency different from Euro (not only U.S. Dollars, but also yen for example). For this reason, the Ferrari should repeat the behaviors assumed in this model for any type of no-Euro credits that it has.

Hereinafter U. S. Dollars are called simply Dollars.

2. The game and stabilizing proposal

2.1. The description of the game

We assume that our **first player** is the Ferrari spa, which chooses to buy Euro futures contracts to hedge against an upwards change of Euro-Dollar exchange rate; the Ferrari should cash a certain quantity of Dollar credits, which represent a quantity M_1 of Euros that it would cash at time 1 with the Euro-Dollar exchange rate of time 0. Therefore, the Ferrari can choose a strategy $x \in [0, 1]$, representing the percentage of the quantity of the total Euros M_1 that the Ferrari itself will purchase through Euro futures, depending on it wants:

1) to not hedge, converting in Euros all the Dollar credits that it will cash at time 1 ($x = 0$);

2) to hedge partially, buying Euro futures for a part of its Dollar credits that it will cash at time 1 and converting in Euros the rest ($0 < x < 1$);

3) to hedge totally, buying Euro futures for all its Dollar credits ($x = 1$).

On the other hand, our **second player** is the Unicredit bank operating on the Euro spot market. The Unicredit works in our game also on the Euro futures market:

1) taking advantage of possible gain opportunities - given by misalignment between Euro spot and futures prices (both expressed in Dollars);

2) or accounting for the loss obtained, because it has to close the position of short sales opened on the Euro spot market.

These actions determine the payoff of the Unicredit. The Unicredit can therefore choose a strategy $y \in [-1, 1]$, which represents the percentage of the quantity of Euros M_2 that it can buy (in algebraic sense) with its financial resources, depending on it intends:

1) to purchase Euros on the spot market ($y > 0$);

2) to short sell Euros on the spot market ($y < 0$);

3) to not intervene on the Euro spot market ($y = 0$).

In Fig. 2, we illustrate the bi-strategy space $E \times F$ of the game.

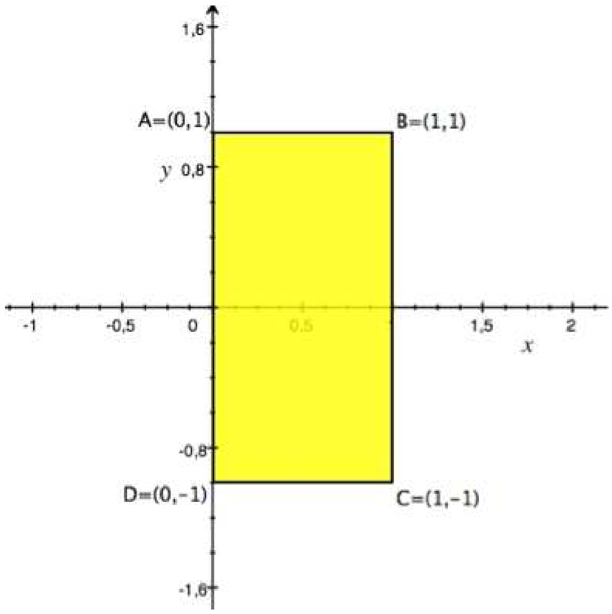

Figure 2: The bi-strategy space of the game

2.2. The payoff function of the Ferrari

The payoff function of the Ferrari, that is the function which represents quantitative relative gain of the Ferrari, referred to time 1, is given by the net gain obtained on not hedged Dollar credits expressed in Euros $x'M_1$ (here $x' := 1 - x$). The gain related with the not hedged Dollar credits is given by the quantity of the not hedged Dollar credits expressed in Euros $(1 - x)M_1$, multiplied by the difference $F_0 - S_1(y)$, between the Euro futures price at time 0 (the term F_0) - which the Ferrari should pay, if it decides to hedge its Dollar credits - and the Euro spot price $S_1(y)$ at time 1, when the Ferrari actually buys Euros converting its Dollar credits that it did not hedge. So, the payoff function of the Ferrari is defined by

$$f_1(x, y) = F_0 M_1 x' - S_1(y) M_1 x' = (F_0 - S_1(y)) M_1 (1 - x), \tag{1}$$

for every bi-strategy (x, y) in $E \times F$, where:

1) M_1 is the amount of Euros that the Ferrari should buy at time 1 converting its Dollar credits by the exchange rate at time 0;

2) $x' = 1 - x$ is the percentage of the Euros that the Ferrari buys on the spot market at time 1, without any hedge (and therefore exposed to the fluctuations of Euro-Dollar exchange rate);

3) F_0 is the Euro futures price (expressed in Dollars) at time 0. It represents the Euro price established at time 0 that the Ferrari has to pay at time 1 in order to buy Euros. By definition, the futures price after $(T - 0)$ time units is given by $F_0 = S_0 u^T$, where $u = 1 + i$ is the (unit) capitalization factor with rate i. By i we mean the risk-free interest rate charged by banks on deposits of other banks, the so-called LIBOR rate. S_0 is, on the other hand, the Euro spot price at time 0. S_0 is constant because it is not influenced by our strategies x and y.

4) $S_1(y)$ is the Euro spot price (expressed in Dollars) at time 1, after that the Unicredit has implemented its strategy y. It is given by $S_1(y) = S_0 u + nuy$, where n is the marginal coefficient representing the effect of the strategy y on the price $S_1(y)$. The price function S_1 depends on y because, if the Unicredit intervenes in the Euro spot market by a strategy y not equal to 0, then the Euro price S_1 changes, since any demand change has an effect on the Euro-Dollar exchange rate. We are assuming linear the dependence $n \mapsto ny$ in S_1. The value S_0 and the value ny should be capitalized, because they should be transferred from time 0 to time 1.

The payoff function of the Ferrari. Therefore, recalling the definitions of F_0 and S_1, the payoff function f_1 of the Ferrari (from now on, the factor nu will be indicated by ν) is given by:

$$f_1(x, y) = -M_1(1 - x)\nu y = -M_1(1 - x)\nu y. \tag{2}$$

2.3. The payoff function of the Unicredit

The payoff function of the Unicredit at time 1, that is the algebraic gain function of the Unicredit at time 1, is the multiplication of the quantity of Euros bought on the spot market, that is yM_2, by the difference between the Euro futures price $F_1(x, y)$ (it is a price established at time 1 but cashed at time 2) transferred to time 1, that is $F_1(x, y)u^{-1}$, and the purchase price of Euros at time 0, say S_0, capitalized at time 1 (in other words we are accounting for all balances at time 1).

2.3.1. Stabilizing strategy of normative authority.

In order to avoid speculations on Euro spot and futures markets by the Unicredit, which in this model is the only one able to determine the Euro spot price (and consequently also the Euro futures price), we propose that the normative authority imposes to the Unicredit the payment of a tax on the sale of the Euro futures. So the Unicredit can't take advantage of swings of Euro-Dollar exchange rate caused by itself. We assume that this tax is fairly equal to the incidence of the strategy of the Unicredit on the Euro spot price, so the price effectively cashed or paid for the Euro futures by the Unicredit is $F_1(x,y)u^{-1} - \nu y$, where νy is the tax paid by the Unicredit, referred to time 1.

Remark. We note that if the Unicredit wins, it acts on the Euro futures market at time 2 in order to cash the win, but also in case of loss it must necessarily act in the Euro futures market and account for its loss because at time 2 (in the Euro futures market) it should close the short-sale position opened on the Euro spot market.

The payoff function of the Unicredit is defined by:

$$f_2(x,y) = yM_2(F_1(x,y)u^{-1} - \nu y - S_0 u), \tag{3}$$

where:

(1) y is the percentage of Euros that the Unicredit purchases or sells on the spot market;

(2) M_2 is the maximum amount of Euros that the Unicredit can buy or sell on the spot market, according to its economic availability;

(3) S_0 is the price (expressed in Dollars) paid by the Unicredit in order to buy Euros. S_0 is a constant because our strategies x and y do not influence it.

(4) νy is the normative tax on the price of the Euro futures paid at time 1. We are assuming that the tax is equal to the incidence of the strategy y of the Unicredit on the Euro price S_1.

(5) $F_1(x,y)$ is the Euro futures price (expressed in Dollars), established at time 1, after the Ferrari has played its strategy x. The function price

F_1 is given by $F_1(x, y) = S_1(y)u + mux$, where $u = 1 + i$ is the factor of capitalization of interests. By i we mean risk-free interest rate charged by banks on deposits of other banks, the so-called LIBOR rate. With m we intend the marginal coefficient that measures the influence of x on $F_1(x, y)$. The function F_1 depends on x because, if the Ferrari buys Euro futures with a strategy $x \neq 0$, the price F_1 changes because an increase of Euro futures demand influences the Euro futures price. The value S_1 should be capitalized because it follows the fundamental relationship between futures and spot prices (see subsection 1.1, no. 7). The value mx is also capitalized because the strategy x is played at time 0 but has effect on the Euro futures price at time 1.

(6) $(1 + i)^{-1}$ is the discount factor. $F_1(x, y)$ must be translated at time 1, because the money for the sale of Euro futures are cashed at time 2.

The payoff function of the Unicredit. Recalling functions F_1 and f_2, we have

$$f_2(x, y) = yM_2mx, \tag{4}$$

for each $(x, y) \in E \times F$.

The payoff function of the game is so given, for every $(x, y) \in E \times F$, by:

$$f(x, y) = (-\nu yM_1(1 - x), yM_2mx). \tag{5}$$

2.4. The payoff functions in presence of collaterals

In this game we don't consider the presence of collateral. But:

- even if the price F_0 will be paid at time 1, the Ferrari could deposit, already at time 0, the sum F_0 as guarantee that (at the expiry) the contract will be respected.
- even if the price F_1 is paid at time 2, the Unicredit could deposit, already at time 1, the sum F_1 as guarantee that (at the expiry) the contract will be respected.

Proposition 1. *Let F_0 be the Euro futures price at time 0 and let $u := (1+i)$ be the capitalization factor. Then, the payoff function ${f^c}_1$ of the Ferrari, in presence of collateral, is the same of the payoff function f_1 of the Ferrari without collateral.*

Proof. In order to calculate the win of the Ferrari at the time 1, we recall its payoff function (see the Eq.(2))

$$f_1(x, y) = -\nu y M_1(1 - x).$$

In presence of collaterals, at the sum F_0 (that is paid as collateral at time 0 and for this reason it has to be capitalized) must be subtracted the interests $F_0 i$, cashed by the Ferrari on the deposit of collateral.

So, in the payoff function f_1 of the Ferrari we have to put the value

$$F_0 u - F_0 i \tag{6}$$

in place of the futures price F_0.

We will show that the value obtained in the Eq. (6) is equal to the value in place of which must be replaced, that is the Euro futures price F_0. So we want show that

$$F_0 u - F_0 i = F_0.$$

Recalling that $u := (1 + i)$, we have

$$F_0(1 + i) - F_0 i = F_0.$$

This completes the proof. ■

Remark. So we have shown that, in presence of collaterals, the payoff function f_1 of the Ferrari that we have found before without considering eventual collateral, results valid also with guarantee deposits.

Proposition 2. *Let*

$$F_1(x, y) = S_1(y)u + mux$$

be the Euro futures price at time 0 and let $u := (1 + i)$ *be the capitalization factor. Then, the payoff function* ${f^c}_2$ *of the Unicredit, in presence of collateral, is the same of the payoff function* f_2 *of the Unicredit without collateral.*

Proof. In order to calculate the win of the Unicredit at the time 1, we recall its payoff function (see the Eq.(4))

$$f_2(x, y) = yM_2mx.$$

In presence of collaterals, at the value F_1 (that is paid as collateral at time 1) we must subtract the interests (actualized at time 1) on the deposit of collateral cashed at time 2 by the Unicredit.

The interests cashed by the Unicredit are given by

$$F_1(x, y)iu^{-1}.$$

So, in the payoff function f_2 of the Unicredit we have to put the value

$$F_1(x, y) - F_1(x, y)iu^{-1} \tag{7}$$

in place of the Euro futures price actualized F_1u^{-1}.

We will show that the value obtained in the Eq. (7) is equal to the value in place of which must be replaced, that is the Euro futures price actualized $F_1(x, y)u^{-1}$. So we want show that

$$F_1(x, y) - F_1(x, y)iu^{-1} = F_1(x, y)u^{-1}.$$

Recalling that

$$F_1(x, y) = S_1(y)u + mux,$$

we obtain

$$S_1(y)u + mux - (S_1(y) + mx)uu^{-1}i = (S_1(y)u + mux)u^{-1},$$

and therefore

$$S_1(y)u + mux - (S_1(y) + mx)i = S_1(y) + mx.$$

Recalling that $u = (1 + i)$, we have

$$S_1(y)(1+i) + mx(1+i) - S_1(y)i + mxi = S_1(y) + mx.$$

This completes the proof. ■

Remark. So we have shown that, in presence of collaterals, the payoff function of the Unicredit that we have found before without considering eventual collateral, results valid also with guarantee deposits.

3. Study of the game

3.1. Critical space of the game

Since we are dealing with a non-linear game it is necessary to study in the bi-win space also the points of the critical zone, which belong to the bi-strategy space. In order to find the critical area of the game we consider the Jacobian matrix and we put its determinant equal 0.

For what concern the gradients of f_1 and f_2, we have

$$\nabla \quad f_1 = (M_1 y\nu, -\nu M_1(1-x))$$
$$\nabla \quad f_2 = (M_2 my, M_2 mx).$$

The determinant of the Jacobian matrix is

$$\det J_{f(x,y)} = M_1 M_2 \nu ymx + M_1 M_2 m(1-x)\nu y.$$

Therefore the critical space of the game is

$$Z_f = \{(x,y) : M_1 M_2 \nu ymx + M_1 M_2 m(1-x)\nu y = 0\}.$$

Dividing by $M_1 M_2 \nu m$, which are all positive numbers (strictly greater than 0), we have:

$$Z_f = \{(x,y) : yx + (1-x)y = 0\}.$$

Finally we have

$$Z_f = \{(x,y) : y = 0\}.$$

The critical area of our bi-strategy space is represented in the figure 3 by the segment $[H, K]$.

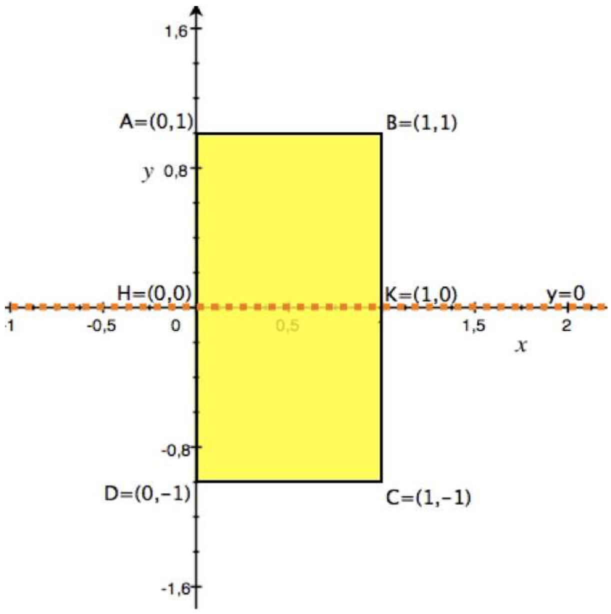

Figure 3: The critical space of the game

3.2. Payoff space

In order to represent graphically the payoff space $f(E \times F)$, we transform, by the function f, all the sides of bi-strategy rectangle $E \times F$ and the critical space Z of the game G.

1) The segment $[A, B]$ is the set of all the bi-strategies (x, y) such that $y = 1$ and $x \in [0, 1]$.

Calculating the image of the generic point $(x, 1)$, we have $f(x, 1) = (M_1[-\nu(1 - x)], M_2 mx)$.

Therefore setting $X = M_1[-\nu(1 - x)]$ and $Y = M_2 mx$, and assuming $M_1 = 1, M_2 = 2$, and $\nu = m = 1/2$, we have $X = -(1/2)(1 - x)$ and $Y = x$.

Replacing Y instead of x, we obtain the image of the segment $[A, B]$, defined as the set of the bi-wins (X, Y) such that $X = -(1/2)(1 - Y) = -1/2 + Y$ and $Y \in [0, 1/2]$.

It is a line segment with extremes $A' = f(A)$ and $B' = f(B)$.

Following the procedure described above for the other side of the bi-strategy rectangle and for the critical space, that are the segments $[B, C]$, $[C, D]$, $[D, A]$ and $[H, K]$, we get the figures 4, 5, 6, 7 and 8 on the payoff space $f(E \times F)$ of our game G.

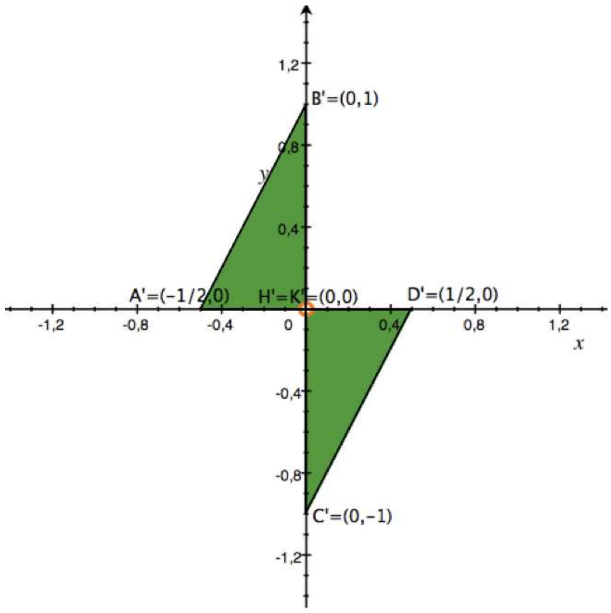

Figure 4: The payoff space of the game G

We can see how the set of possible winning combinations of the two players took a curious butterfly shape that promises the game particularly interesting.

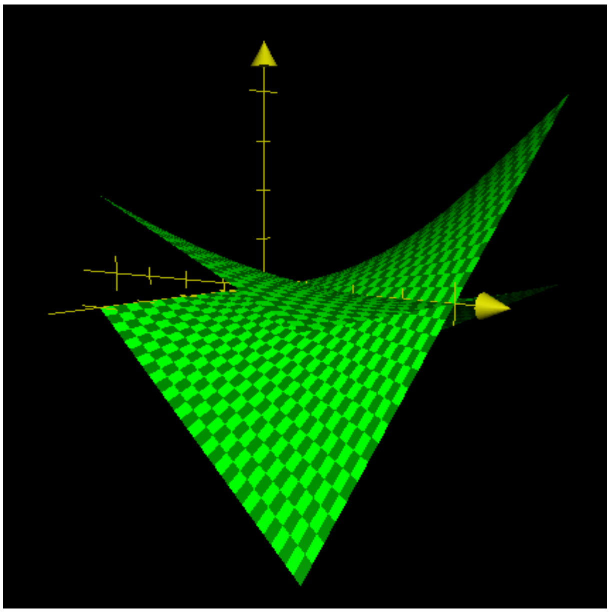

Figure 5: The payoff space of the game G

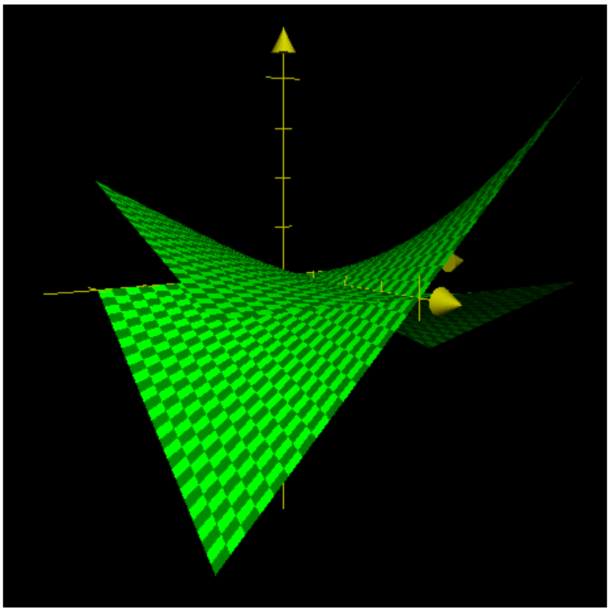

Figure 6: The payoff space of the game G

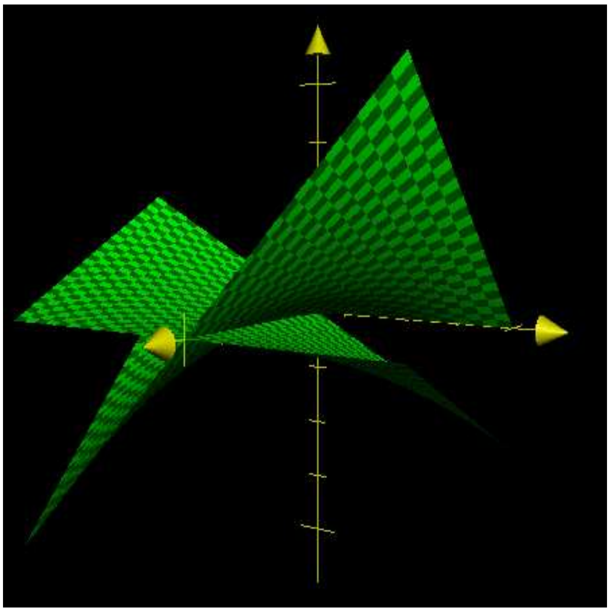

Figure 7: The payoff space of the game G

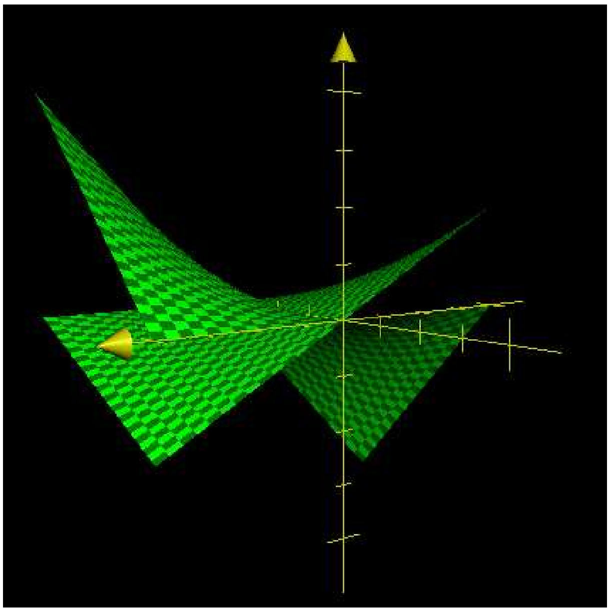

Figure 8: The payoff space of the game G

4. Study of the game and equilibria

4.1. Friendly phase

The superior extremum of the game, that is the bi-win $\alpha = (1/2, 1)$, is a shadow maximum because it doesn't belong to the payoff space:

$$\alpha = (1/2, 1) \notin f(E \times F).$$

The infimum of the game, that is the bi-win $\beta = (-1/2, -1)$, is a shadow minimum because it doesn't belong to the payoff space:

$$\beta = (-1/2, -1) \notin f(E \times F).$$

The weak maximal Pareto boundary of the payoff space is $[B'K'] \cup [H'D']$. The weak maximal Pareto boundary of the bi-strategic space is the retro-image of the weak maximal Pareto boundary of the payoff space, is $[BK] \cup [HD] \cup [HK]$.

The proper maximal Pareto boundary of the payoff space is represented by $\partial^* f(E \times F) = \{B', D'\}$. The proper maximal Pareto boundary of the bi-strategic space is the reciprocal image of the proper maximal Pareto boundary of the payoff space, is $\partial^* f(E \times F) = \{B, D\}$.

The weak minimal Pareto boundary of the payoff space is $[A'H'] \cup [K'C']$. The weak minimal Pareto boundary of the bi-strategy space is the reciprocal image of the weak minimal Pareto boundary of the payoff space, is $[AH] \cup [KC] \cup [HK]$.

The proper minimal Pareto boundary of the payoff space is represented by $\partial_* f(E \times F) = \{A', C'\}$. The proper minimal Pareto boundary of the bi-strategy space is the reciprocal image of the proper minimal Pareto boundary of the payoff space, is $\partial_* f(E \times F) = \{A, C\}$.

In the figure 9 we show graphically the previous considerations.

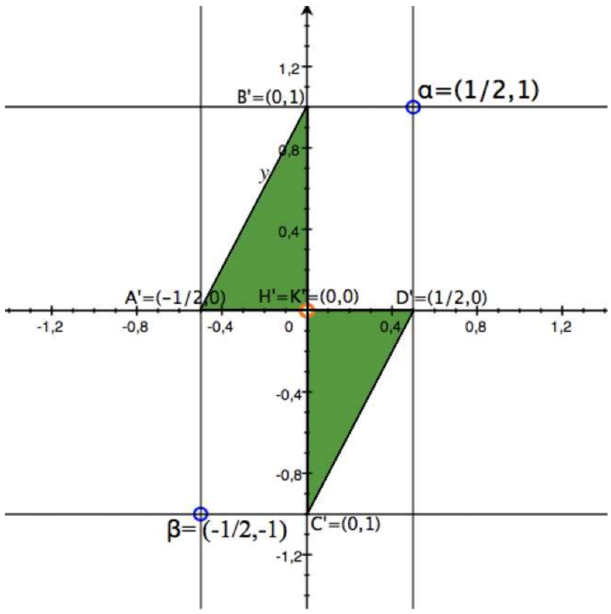

Figure 9: Pareto boundaries and extrema of the game

Control and accessibility of non-cooperative Pareto boundaries. *Definition of Pareto control.* The Ferrari can cause a Pareto bi-strategy x_0 if exists a strategy such that for every strategy y of the Unicredit the pair (x_0, y) is a Pareto pair.

In this regard, in our game there are no maximal Pareto controls, nor minimal. So neither player can decide to go on the Pareto boundary without cooperation with the other one. The game promises to be quite complex to resolve in a satisfactory way for both players.

4.2. Nash equilibria

If the two players decide to adopt a selfish behavior, they choose their own strategy maximizing their partial gain. In this case, we should consider the classic Nash best reply correspondences.

The best reply correspondence of the Ferrari is the correspondence $B_1 : F \to E$ given by $y \mapsto \max_{f_1(\cdot,y)} E$, where $\max_{f_1(\cdot,y)} E$ is the set of all strategies in E which maximize the section $f_1(\cdot, y)$.

Symmetrically, the best reply correspondence $B_2 : E \to F$ of the Unicredit is given by $x \mapsto \max_{f_2(x,\cdot)} F$.

Choosing $M_1 = 1$, $\nu = 1/2$, $M_2 = 2$ and $m = 1/2$, which are positive numbers (strictly greater than 0), and recalling that $f_1(x, y) = -M_1\nu y(1-x)$, we have $\partial_1 f_1(x, y) = M_1\nu y$, this derivative has the same sign of y, and so:

$$B_1(y) = \begin{cases} \{1\} & \text{if} \quad y > 0 \\ E & \text{if} \quad y = 0 \\ \{0\} & \text{if} \quad y < 0 \end{cases}.$$

Recalling that $f_2(x, y) = M_2mxy$, we have $\partial_2 f_2(x, y) = M_2mx$ and so: $B_2(x) = \{1\}$ if $x > 0$ and $B_2(x) = F$ if $x = 0$.

In Fig.10 we have in red the inverse graph of B_1, and in blue that one of B_2.

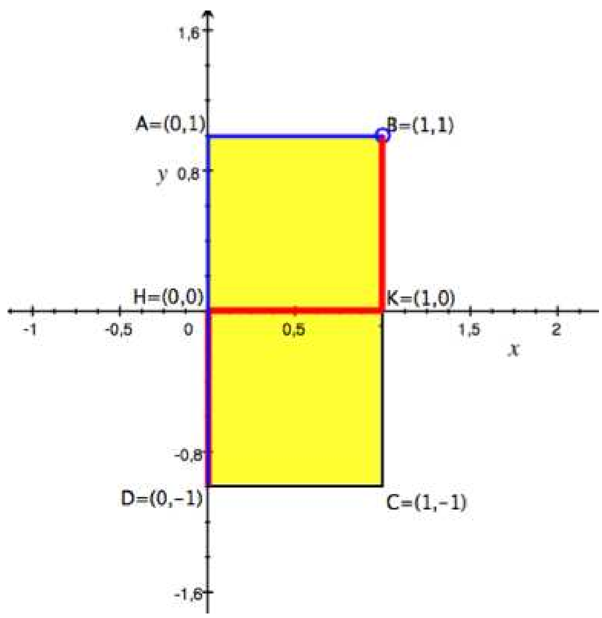

Figure 10: Nash equilibria

The set of Nash equilibria, that is the intersection of the two best reply graphs (graph of B_2 and the symmetric of B_1), is $\{(1,1)\} \cup [H,D]$.

Analysis of Nash equilibria. The Nash equilibria can be considered quite good, because they are on the weak maximal Pareto boundary. It is clear that if the two players pursue the profit, and choose their selfish strategies to obtain the maximum possible win, they arrive on the weak maximal boundary. The selfishness, in this case, pays well. This purely mechanical examination, however, leaves us unsatisfied. The Ferrari has two Nash possible alternatives: not to hedge, playing 0, or to hedge totally, playing 1. Playing 0 it could both to win or lose, depending on the strategy played by the Unicredit; opting instead for 1, the Ferrari guarantee to himself to leave the game without any loss and without any win.

Analysis of possible Nash strategies. If the Ferrari adopts a strategy $x \neq 0$, the Unicredit plays the strategy 1 winning something, or else if the Ferrari plays 0 the Unicredit can play all its strategy set F, indiscriminately, without obtaining any win or loss. These considerations lead us to believe that the Unicredit will play 1, in order to try to win at least "something", because if the Ferrari plays 0, its strategy y does not affect its win. The Ferrari, which knows that the Unicredit very likely chooses the strategy 1, will hedge playing the strategy 1. So, despite the Nash equilibria are infinite, it is likely the two players arrive in $B = (1,1)$, which is part of the proper maximal Pareto boundary. Nash is a viable, feasible and satisfactory solution, at least for one of two players, presumably the Unicredit.

4.3. Defensive phase

We suppose that the two players are aware of the will of the other one to destroy it economically, or are by their nature cautious, fearful, paranoid, pessimistic or risk averse, and then they choose the strategy that allows them to minimize their loss. In this case, we talk about defensive strategies.

Conservative value and meetings. *Conservative value of a player.* It is defined as the maximization of its function of worst win. Therefore, the conservative value of the Ferrari is $v_1^\sharp = \sup_{x \in E} f_1^\sharp(x)$, where $f_1^\sharp$ is the function of worst win of the Ferrari, and it is given by $f_1^\sharp(x) = \inf_{y \in F} f_1(x,y)$, for every x in E.

Recalling the Eq. (2), that is $f_1(x, y) = M_1[-\nu y(1-x)]$, and choosing $M_1 = 1$, $\nu = 0.5$, $M_2 = 2$ and $m = 0.5$, which are always positive numbers (strictly greater than 0), we have:

$$f_1^\sharp = \inf_{y \in F} M_1[-\nu y(1-x)].$$

Therefore since the offensive strategies of the Unicredit are $O_2(x) = \begin{cases} \{1\} & \text{if} \quad 0 \leq x < 1 \\ \{F\} & \text{if} \quad x = 1 \end{cases}$, we obtain:

$$f_1^\sharp(x) = \begin{cases} \{M_1[-\nu(1-x)]\} & \text{if} \quad 0 \leq x < 1 \\ \{0\} & \text{if} \quad x = 1 \end{cases}.$$

In the figure 11 $f_1^\sharp$ appears graphically.

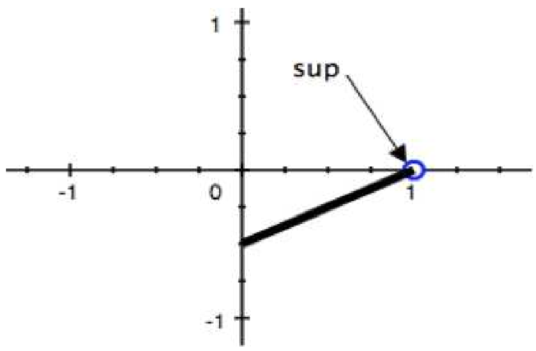

Figure 11: Graphical representation of $f_1^\sharp$, the function of worst win of the Ferrari.

So the defense (or conservative) strategy of the Ferrari is given by

$$x_\sharp = 1$$

and the conservative value of the Ferrari is

$$v_1^\sharp = \sup_{x \in E} \inf_{y \in F} M_1[-\nu y(1-x)] = 0. \tag{8}$$

On the other hand, the conservative value of the Unicredit is given by $v_2^\sharp = \sup_{y \in F} f_2^\sharp$, where $f_2^\sharp$ is the function of the worst win of the Unicredit. It is given by $f_2^\sharp(y) = \inf_{x \in E} f_2(x, y)$, for every $y \in F$.

Recalling the Eq. (4), that is

$$f_2(x, y) = M_2 mxy,$$

and choosing $M_1 = 1, \nu = 0.5, M_2 = 2$ and $m = 0.5$, which are always positive numbers (strictly greater than 0), we have:

$$f_2^\sharp = \inf_{x \in E} M_2 mxy.$$

Therefore since the offensive strategies of the Ferrari are $O_1(y) = \begin{cases} \{0\} & \text{if} \quad y > 0 \\ \{E\} & \text{if} \quad y = 0 \\ \{1\} & \text{if} \quad y < 0 \end{cases}$,

we obtain:

$$f_2^\sharp(y) = \begin{cases} \{0\} & \text{if} \quad y \geq 0 \\ \{M_2 my\} & \text{if} \quad y < 0 \end{cases}.$$

In the figure 12 $f_2^\sharp(y)$ appears graphically.

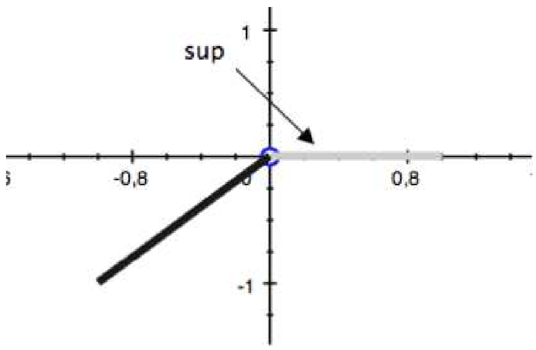

Figure 12: Graphical representation of $f_2^\sharp$, the function of worst win of the Unicredit.

So the defense (or conservative) strategy of the Unicredit is given by

$$y_\sharp = [0, 1]$$

and the conservative value of the Unicredit is

$$v_2^\sharp = \sup_{y \in F} \inf_{x \in E} M_2 mxy = 0. \tag{9}$$

Therefore the conservative bi-value is

$$v_f^\sharp = (v_1^\sharp, v_2^\sharp) = (0, 0).$$

Conservative meetings. They are represented by the bi-strategies $(x_\sharp, y_\sharp)$, that are represented by the whole segment $[B, K]$. If the Ferrari and the Unicredit decides to defend themselves against any opponent's offensive strategies, they arrive on the payoffs subset $[B', K']$, which is part of the weak maximal Pareto boundary. B' is even a point on the proper maximal boundary, while K' is also part of the weak minimal one. In this simplified model, although there is the possibility that the Unicredit decides not to act on the market, obtaining in this way no profit and arriving in K', the Unicredit presumably will choose the defensive strategy $y_\sharp = 1$, because it's the only one that allows him to obtain the maximum possible profit (being able anyway not to incur losses). In this case the players arrive in B', the optimal solution for the Unicredit. This happens because the Ferrari was unable with its strategies $x \in [0, 1]$ to lead to a lowering of the Euro futures price.

Remark. In reality, however, in addiction to the Ferrari there are other traders, which could also cause a fall in futures prices and then, if the Unicredit would choose a defensive strategy, presumably it would decide to not act on the market with $y_\sharp = 0$. In this case, the conservative meeting would be only one, i.e. $K = (1, 0)$.

4.3.1. Core and conservative parts of the game

Core of the payoff space. The core is the part of the maximal Pareto boundary contained in the upper cone of the payoff

$$v_f^\sharp = (v_1^\sharp, v_2^\sharp) = (0, 0).$$

Therefore we have

$$core'(G) = [B'K'] \cup [H'D'],$$

whose reciprocal image is

$$core(G) = [BK] \cup [HD] \cup [HK].$$

In the figure 13 we can see graphically in red the part of the payoff space where the Ferrari would has a win greater than its conservative value $v_1^\sharp = 0$

(x-axis in pink). On the other hand, in blue is shown the part of the payoff space where the Unicredit obtains a win higher than its conservative value $v_2^\sharp = 0$ (y-axis in blue).

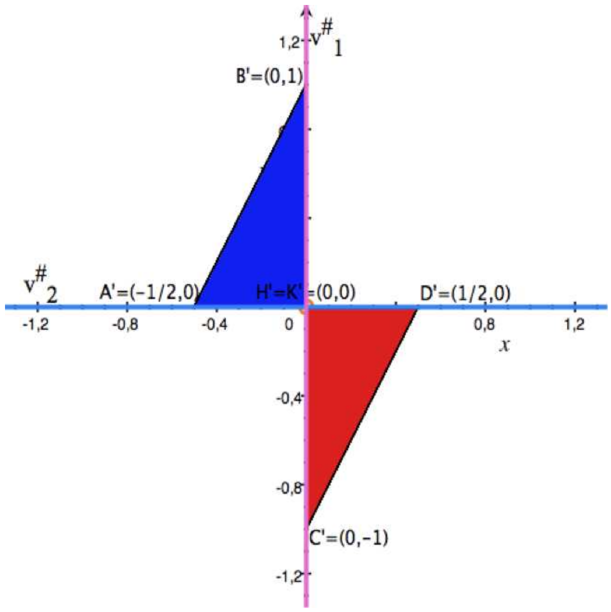

Figure 13: Core and conservative parts on the payoff space.

We note that if both players choose their conservative strategies $x_\sharp = 1$ e $y_\sharp = [0, 1]$, the Ferrari avoids to lose more of its conservative value $v_1^\sharp = 0$ but is automatically unable to get also higher wins. The same discourse does not apply to the Unicredit that may arrive on the segment $[B'K']$. The game is in substance blocked for the Ferrari, that is clearly disadvantaged in respect of the Unicredit.

Remark. Recalling the previous remark (see the previous page 12), the game would be blocked for both, with the Unicredit also unable to get higher wins to its conservative value $v_2^\sharp = 0$ if it decides to play its defensive strategy $y_\sharp = 0$.

Conservative part of the game on the bi-strategy space. It is the set of the pairs (x, y) such that

$$f_1(x) \geq v_1^\sharp \wedge f_2(y) \geq v_2^\sharp.$$

Recalling the Eq. (2), that is

$$f_1(x, y) = M_1[-\nu y(1 - x)],$$

and the Eq. (8), that is $v_1^\sharp = 0$, the conservative part of the Ferrari on the bi-strategy space is given by

$$(E \times F)_1^\sharp = M_1[-\nu y(1 - x)] \geq 0,$$

which developed becomes

$$-\nu M_1 y \leq 0 \vee x \leq 1 \quad \text{or} \quad -\nu M_1 y \geq 0 \vee x \geq 1.$$

Choosing $M_1 = 1$ and $\nu = 0.5$, which are always positive numbers (strictly greater than 0), we obtain the figure 14.

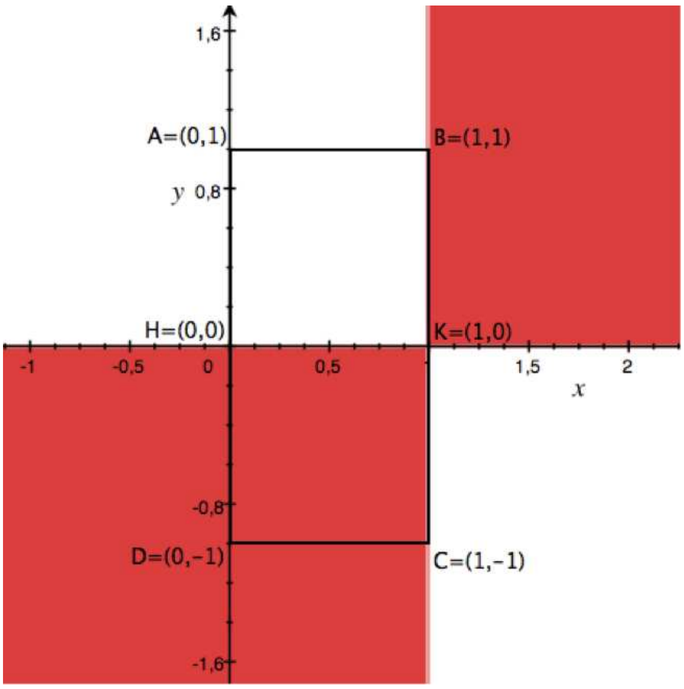

Figure 14: Conservative part of the Ferrari (in red) on the bi-strategy space.

Now talk about the Unicredit. Recalling the Eq. (4), that is

$$f_2(x,y) = M_2 mxy,$$

and the Eq. (9), that is $v_2^\sharp = 0$, the conservative part of the Unicredit on the bi-strategy space is given by

$$(E \times F)_2^\sharp = M_2 mxy \geq 0.$$

Choosing $M_2 = 2$ and $m = 0.5$, which are always positive numbers (strictly greater than 0), we obtain the figure 15.

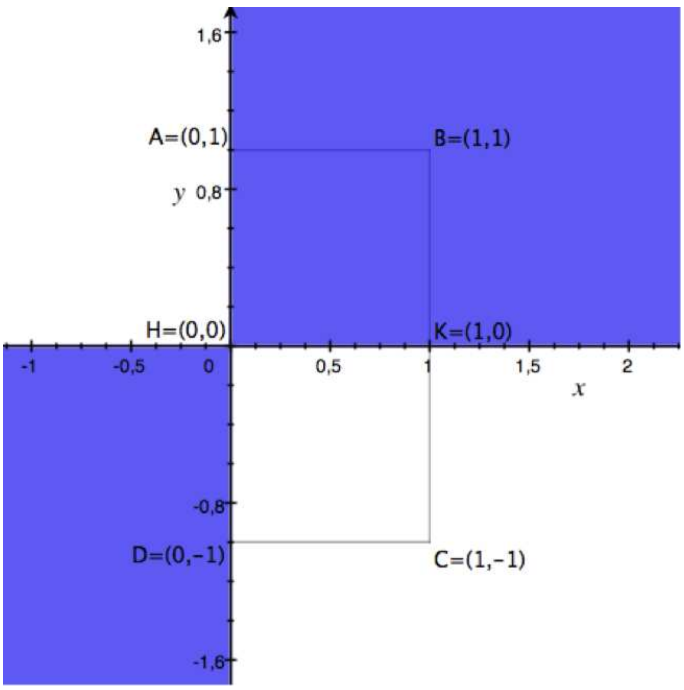

Figure 15: Conservative part of the Unicredit (in light blue) on the bi-strategy space.

Then intersecting the graph of the conservative part (we are talking about the bi-strategy space) of the Ferrari (player 1) and the conservative part of the Unicredit (player 2), we have the conservative part of the game in the bi-strategy space.

It is given by the intersection

$$(E \times F)^\sharp = (E \times F)_1^\sharp \wedge (E \times F)_2^\sharp,$$

and, then

$$(E \times F)^\sharp = M_1[-\nu y(1-x)] \geq 0 \wedge M_2 mxy \geq 0.$$

We observe the graphical result in the figure 16, where the conservative part is easily seen to be a union of three line segments (shown in yellow); this situation was, in any case, quite evident also from the analysis of the figure 13 (representing the transformation of the Core of the game and the conservative parts in the payoff space).

We remark, moreover, that this conservative part coincides with the weak Pareto boundary of the game, that is the set of all bi-strategies which are not strongly dominated by other bi-strategies of the game: $\partial^*_w G$ ={(x,y): does not exist (u, v) in $E \times F$ such that $f(x, y) << f(u, v)$}, where $w << w'$ means that both components of w are strictly less than the corresponding components of w'.

Let us present, now, the figure 16.

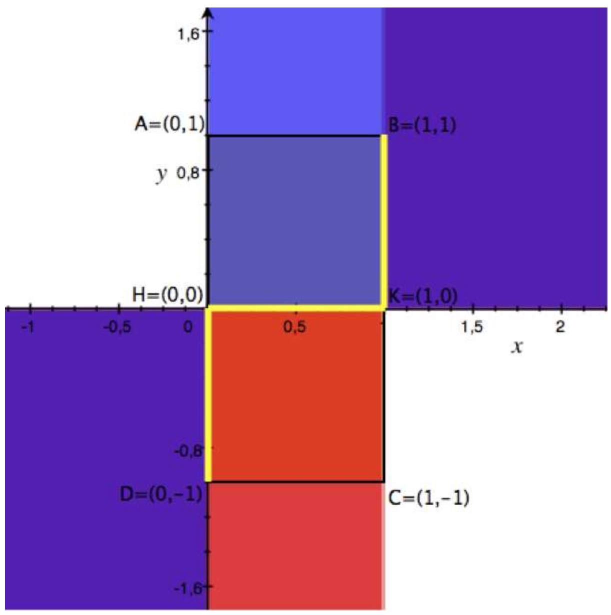

Figure 16: Conservative part of the game (in yellow) on the bi-strategy space.

We see easily that the conservative part of the game, on the bi-strategy space, is given by

$$(E \times F)^\sharp = [BK] \cup [KH] \cup [HD].$$

4.3.2. Conservative knots of the game

Conservative knots. They are, by definition, the strategy pairs (x, y) such that

$$f_1(x, y) = v_1^\sharp \ \ \textit{and} \ \ f_2(x, y) = v_2^\sharp,$$

that is those bi-strategies whose images coincide with the conservative bi-value.

And therefore, recalling the Eq. (2), that is

$$f_1(x, y) = M_1[-\nu y(1 - x)],$$

and the Eq. (8), that is $v_1^\sharp = 0$, any conservative knot verifies the equation:

$$M_1[-\nu y(1 - x)] = 0.$$

Solving the equation, we obtain $M_1 \nu y = 0$ and $1 - x = 0$.

Choosing M_1, e ν, which are always positive numbers (strictly greater than 0), we have:

$$y = 0 \ \ \text{or} \ \ x = 1.$$

Recalling also the Eq. (4), that is

$$f_2(x, y) = M_2 mxy,$$

and the Eq. (9), that is $v_2^\sharp = 0$, we have:

$$M_2 mxy = 0.$$

Choosing M_2 and m, which are always positive numbers (strictly greater than 0), we have:

$$x = 0 \ \ \text{or} \ \ y = 0.$$

Therefore, as we can see in the figure 17, every point $(x, 0)$ of the bi-strategy space, i.e. the segment $[H, K]$, is a conservative knot.

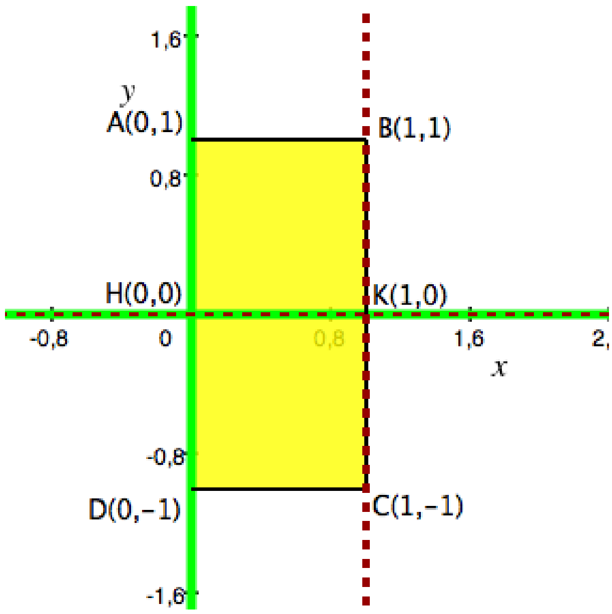

Figure 17: Conservative knots

4.4. Offensive equilibria

If the two players want to think only to ruin the other one, would choose the strategy that makes maximum the loss of the other one. In this case it is nec-essary to talk about multifunction of worst offense.

The multifunction of worst offense of the Ferrari against the Unicredit is the correspondence

$$O_1 : F \rightarrow E : y \mapsto \min_{f_2(\cdot,y)} E$$

where $\min_{f_2(\cdot,y)}$ is the set of all strategies in E that minimize the section $f_2(\cdot, y))$.

On the other hand, the multifunction of worst offense of the Unicredit against the Ferrari is:

$$O_2 : E \rightarrow F : x \mapsto \min_{f_1(x,\cdot)} F.$$

In practice, in order to find O_1 we try the value of x that minimizes f_2; in order to find O_2 we try the value of y that minimize f_1.

Recalling the Eq. (2), that is

$$f_1(x, y) = M_1[-\nu y(1 - x)],$$

we have

$$O_2(x) = \begin{cases} \{1\} & \text{if} \quad 0 \leq x < 1 \\ \{F\} & \text{if} \quad x = 1 \end{cases}.$$

Recalling also the Eq. (4), that is

$$f_2(x, y) = M_2 mxy,$$

we have

$$O_1(y) = \begin{cases} \{0\} & \text{if} \quad y > 0 \\ \{E\} & \text{if} \quad y = 0 \\ \{1\} & \text{if} \quad y < 0 \end{cases}.$$

We observe in the figure 18 the graphs of O_2 (in blue) and of O_1 (in red).

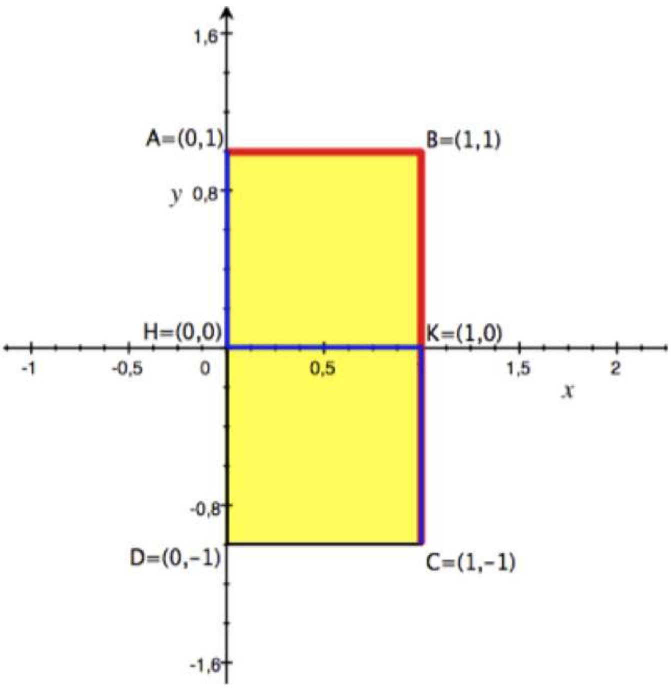

Figure 18: Offensive equilibria

The set of offensive equilibria, that is the intersection of the two worst offense graphs (graph of O_2 and the symmetric of O_1), is

$$Eq(O_1, O_2) = \{(0, 1)\} \cup [KC].$$

Analysis of offensive equilibria. The offensive equilibria may be considered bad because they are on the weak minimal Pareto boundary (indeed the point K' is also part of the weak maximal boundary). In addition, among the offensive equilibria there are also the two points that represent the proper minimal Pareto boundary, i.e. $\{A', C'\}$. It is clear that if the two players want to attack the other one, and decide to choose their strategy just to spite the other player, they arrive on the weak minimal Pareto boundary.

Analysis of possible offensive strategies. Probably the Unicredit plays the strategy $y = 1$ because it is the only one able to maximize the damage of the Ferrari if it plays $x \neq 1$, while if the Ferrari chooses the strategy $x = 1$, the choice of strategy by the Unicredit is indifferent about the damage (zero) procured to the Ferrari.

On the other hand, knowing that the Unicredit chooses the strategy $y = 1$ to try to hurt it, the Ferrari most likely chooses $x = 0$ to be sure that the Unicredit gets the minimum possible win (which, in this case, is equal to 0).

So, despite the offensive equilibria are infinite, the two players most likely arrive in $A = (0, 1)$, which is on the proper minimal Pareto boundary: the offensive strategies of both players can be considered a credible threat. We want to highlight as very likely even if the Ferrari plays its offensive strategies, in our game, however, the Unicredit will not lose.

4.5. Equilibria of devotion

In the event that the two players wanted to "do good" to the other one, they would choose its strategy that maximizes the payoff of the other one. In this case is necessary to talk about multifunction of devotion.

The multifunction of devotion of the Ferrari is the correspondence

$$L_1 : F \to E : y \mapsto \max_{f_2(\cdot, y)} E,$$

where $\max_{f_2(\cdot,y)}$ is the set of all strategies of the Ferrari that maximize the section $f_2(\cdot, y)$).

Symmetrically, the multifunction of devotion $L_2 : E \to F$ of the Unicredit is given by $x \mapsto \max_{f_1(x,\cdot)} F$.

In practice, in order to find L_1 we try the value of x that maximizes f_2; in order to find L_2 we try the value of y that maximize f_1.

Choosing $M_1 = 1$ and $\nu = 0.5$, which are always positive numbers (strictly greater than 0) and recalling the Eq. (2), that is

$$f_1(x, y) = M_1[-\nu y(1 - x)],$$

we have:

$$L_2(x) = \begin{cases} \{-1\} & \text{if} \quad 0 \leq x < 1 \\ \{F\} & \text{if} \quad x = 1 \end{cases}.$$

Recalling also the Eq. (4), that is

$$f_2(x, y) = M_2 mxy,$$

and choosing $M_2 = 2$ and $m = 0.5$, which are always positive numbers (strictly greater than 0), we have

$$L_1(y) = \begin{cases} \{1\} & \text{if} \quad y > 0 \\ \{E\} & \text{if} \quad y = 0 \\ \{0\} & \text{if} \quad y < 0 \end{cases}.$$

In the figure 19 we illustrate in red the inverse graph of $L_1(y)$ and in blue that one of $L_2(x)$.

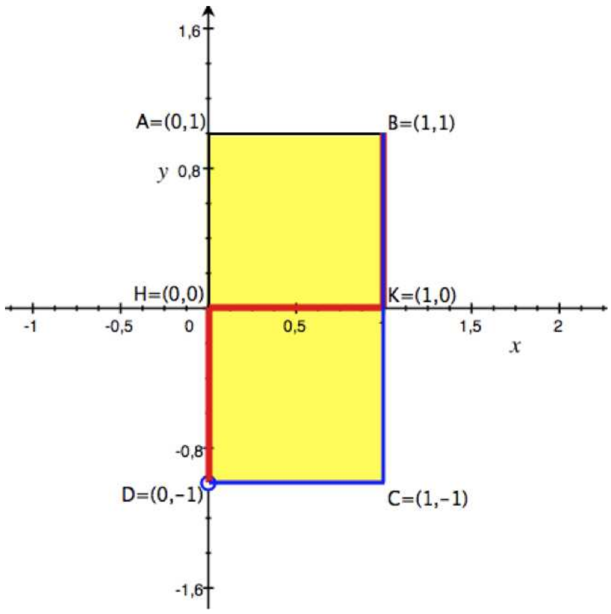

Figure 19: Equilibria of devotion

The set of equilibria of devotion is

$$Eq(L_1, L_2) = \{(0, -1)\} \cup [BK].$$

Analysis of devotion equilibria. The equilibria of devotion can be considered good because they are on the weak maximal Pareto boundary (indeed the point K' is also part of the weak minimal boundary). Also among the devote equilibria there are even the two the points that represent the proper maximal Pareto boundary, i.e. $\{B', D'\}$.

It is clear that if both players ignore their good and decide to choose their strategy selflessly so that the other one has the maximum possible win, they arrive on the weak maximal Pareto boundary.

Analysis of possible devotion strategies. The Unicredit probably plays the strategy $y = -1$ because it is the only one able to maximize the win of the Ferrari if it plays $x \neq 1$, while if the Ferrari chooses the strategy $x = 1$, the choice of strategy of the Unicredit is indifferent about the win (equal to 0) of the Ferrari.

On the other hand, the Ferrari, knowing that the Unicredit chooses the strategy $y = -1$ in order to help it, most likely chooses $x = 0$. So the Unicredit gets the highest possible win, which in this case is equal to 0. We can see that although the equilibria of devotion are infinite, the two players most likely arrive in $D = (0, -1)$, which is on the proper maximal Pareto boundary.

In case of devote strategies adopted by the Unicredit, most likely the Ferrari manages to win the maximum possible sum, while it is not the same for the Unicredit.

4.6. Cooperative solutions

The best way for the two players to get both a gain is to find a cooperative solution. One way would be to divide the **maximum collective profit**, determined by the maximum of the collective gain functional g, defined by $g(X, Y) = X + Y$, on the payoffs space of the game G, i.e the profit $W = \max_{f(E\times F)} g$. The maximum collective profit W is attained at the point B', which is the only bi-win belonging to the straight line $g^{-1}(1)$ (with equation $g = 1$) and to the payoff space $f(E \times F)$. So, the Ferrari and the Unicredit play $(1, 1)$, in order to arrive at the payoff B'. Then, they split the obtained bi-gain B' by means of a contract.

Financial point of view. The Ferrari buys futures to create artificially a misalignment between futures and spot prices; misalignment that is exploited by the Unicredit, which get the maximum win $W = 1$.

For a **possible fair division** of $W = 1$, we employ a *transferable utility solution*: finding on the transferable utility Pareto boundary of the payoff space a non-standard Kalai-Smorodinsky solution (non-standard because we do not consider the whole game, but only its maximal Pareto boundary).

We find the supremum of maximal boundary,

$$\sup \partial^* f(E \times F),$$

which is the point $\alpha = (1/2, 1)$, and we join it with the infimum of maximal Pareto boundary,

$$\inf \partial^* f(E \times F),$$

which is $(0, 0)$.

We note that the infimum of our maximal Pareto boundary is equal to $v^\sharp = (0, 0)$ (the conservative bi-gain of the game). The intersection point P, between the straight line of maximum collective win (i.e. $(g = 1)$) and the straight line joining the supremum of the maximal Pareto boundary with its infimum (i.e., the line $Y = 2X$) is the desirable division of the maximum collective win $W = 1$ between the two players. The figure 20 shows the situation.

The point $P = (1/3, 2/3)$ suggests that the Ferrari should receive $1/3$, by contract, from the Unicredit, while at the Unicredit remains the win $2/3$.

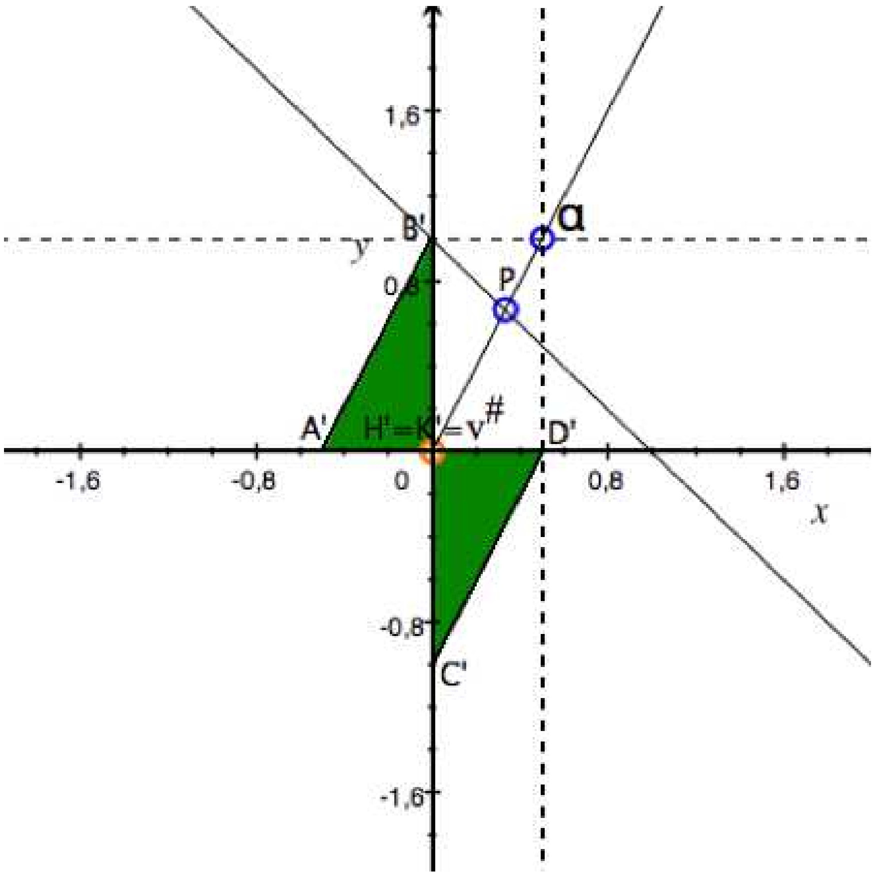

Figure 20: Transferable utility solution: cooperative solution

5. Conclusions

The games just studied suggests a possible regulatory model providing the stabilization of the currency market through the introduction of a tax on currency transactions. In fact, in this way, it could be possible to avoid speculative attacks against the Euro, speculative attacks which constantly affect modern economy. The Unicredit could equally gains without burdening on the financial system by unilateral manipulations of currency exchange rate.

The unique optimal solution is the cooperative one above exposed, otherwise the game appears like a sort of “your death, my life”. This type of situation happens often in the economic competition and leaves no escapes if either player decides to work alone, without a mutual collaboration. In fact, all non-cooperative solutions lead dramatically to mediocre results for at least one of the two players.

Now it is possible to provide an interesting key in order to understand the conclusions which we reached using the transferable utility solution. Since the point $B = (1, 1)$ is also the most likely Nash equilibrium, the number $1/3$ (that the Unicredit pays by contract to the Ferrari) can be seen as the fair price paid by the Unicredit to be sure that the Ferrari chooses the strategy $x = 1$, so they arrive effectively to more likely Nash equilibrium $B = (1, 1)$, which is also the optimal solution for the Unicredit.

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